ME 6135: Advanced Aerodynamics

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Lecture-10

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Classical Thin Airfoil Theory

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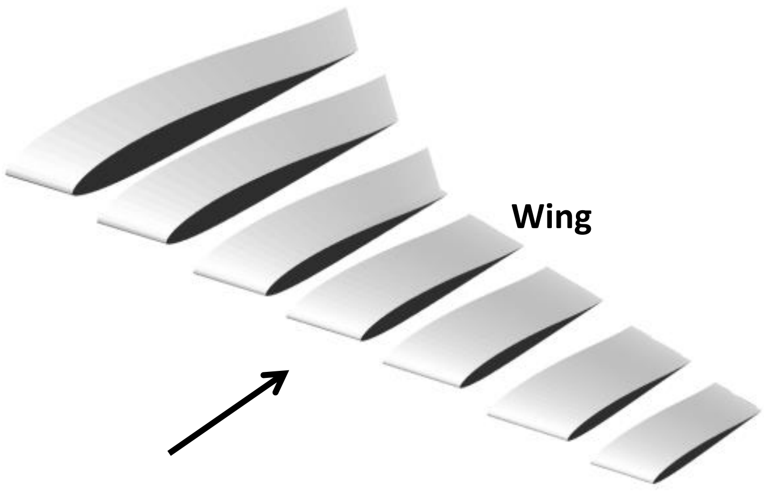
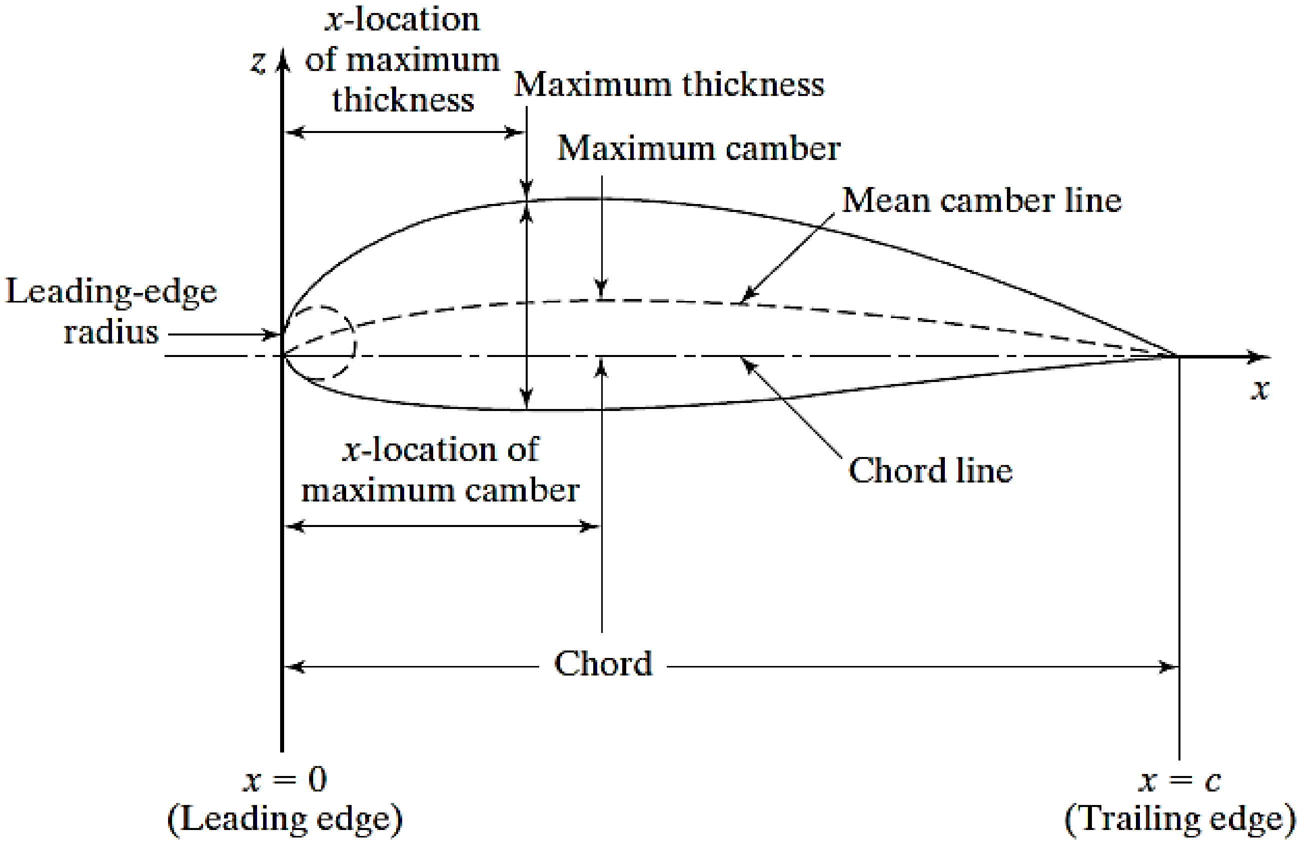
Airfoil Nomenclature

The straight line connecting the leading edge (L.E.) and trailing edge (T.E.) is the **chord line**.

Mean camber line is the locus of points halfway between the upper and lower surfaces as measured perpendicular to the mean camber line itself.

The **camber** is the maximum distance between the mean camber line and the chord line, measured perpendicular to the chord line.



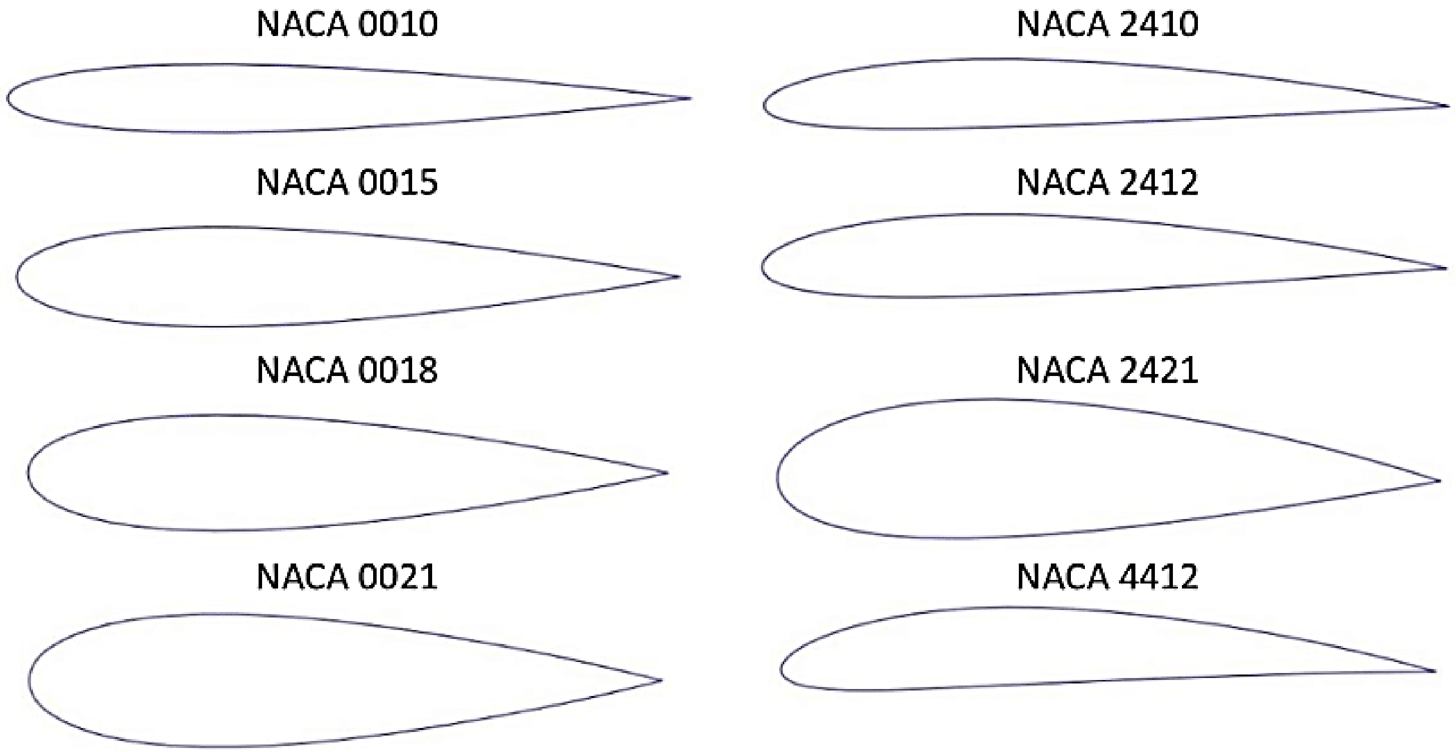


Airfoil

Airfoils


Symmetrical airfoil


Cambered airfoil


NACA 0010
NACA 2410
NACA 0015
NACA 2412
NACA 0018
NACA 2421
NACA 0021
NACA 4412

Airfoil Nomenclature

NACA airfoils

(**N**ational **A**dvisory **C**ommittee for **A**eronautics)

NASA (**N**ational **A**eronautics and **S**pace **A**dministration)

NACA X X X X

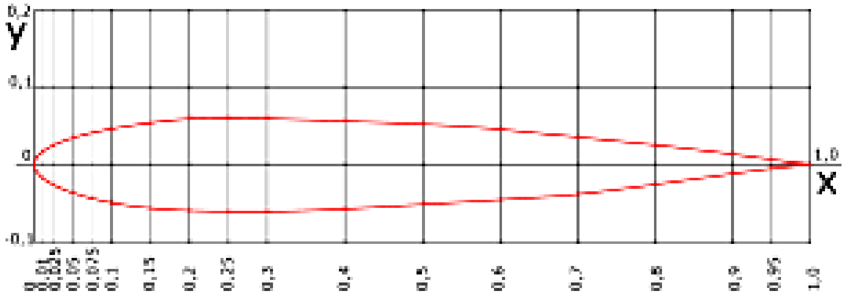
- Maximum thickness in hundredths of chord
- Location of maximum camber from LE in tenths of chord
- Maximum camber in hundredths of chord

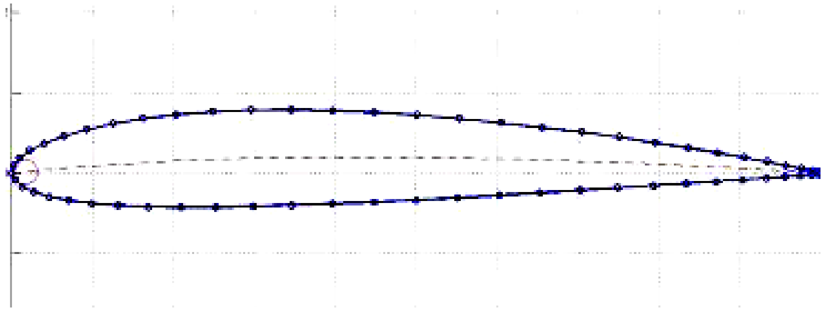
NACA 0 0 12

- Maximum thickness 12%*c*
- 0; No camber
- Zero camber (symmetric airfoil)

NACA 2 4 12

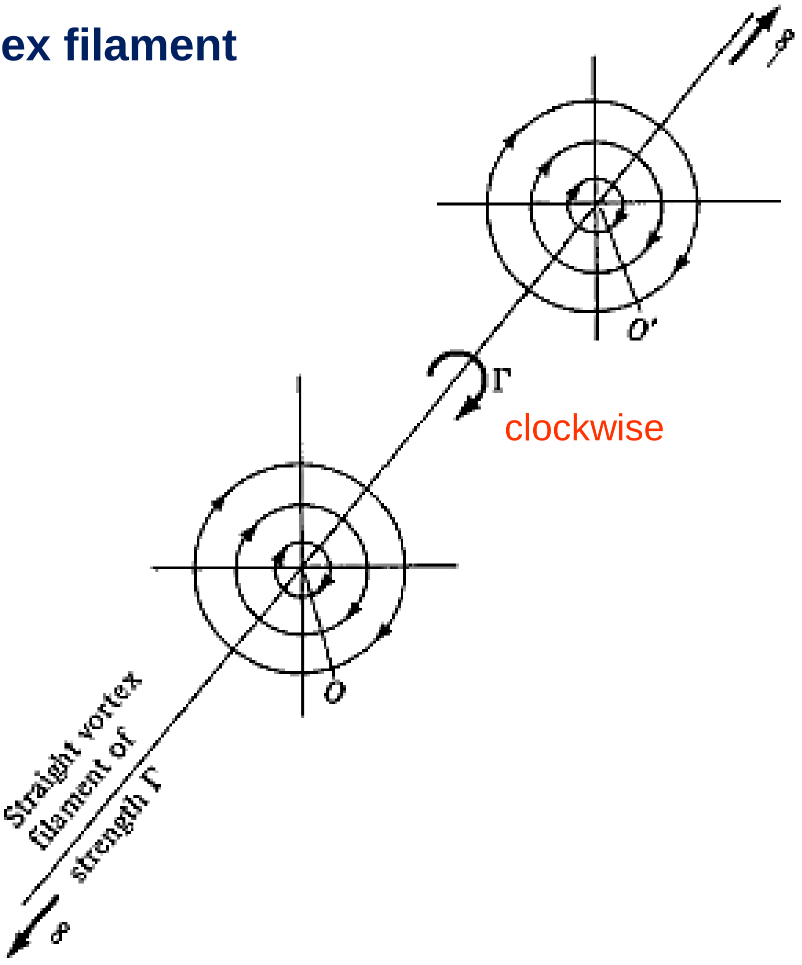
- Maximum thickness 12%*c*
- Maximum camber is at 0.4c from LE
- Maximum camber 2%c (asymmetric airfoil)





Low Speed Flow Over Airfoils

Vortex filament

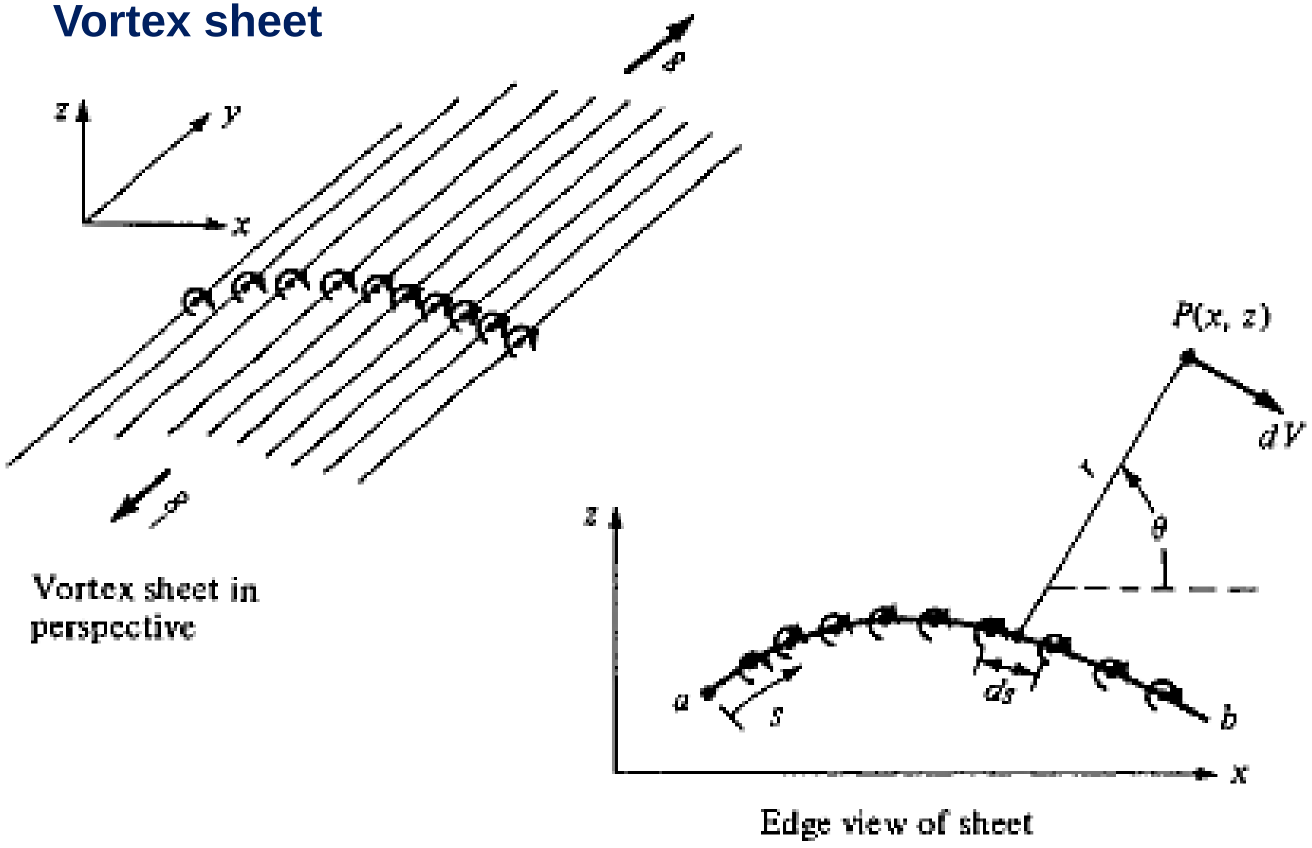


clockwise

Figure 4.7 Vortex filament.

Infinite no. of point vortices along a straight line extending to infinity (+∞ to - ∞)

Vortex sheet



Vortex sheet in perspective

Edge view of sheet

Figure 4.8 Vortex sheet.

Infinite no. of vortex filaments side by side, where strength of each filament is infinitesimally small.

Induced velocity, dV due to infinitesimal vortex filament

Low Speed Flow Over Airfoils

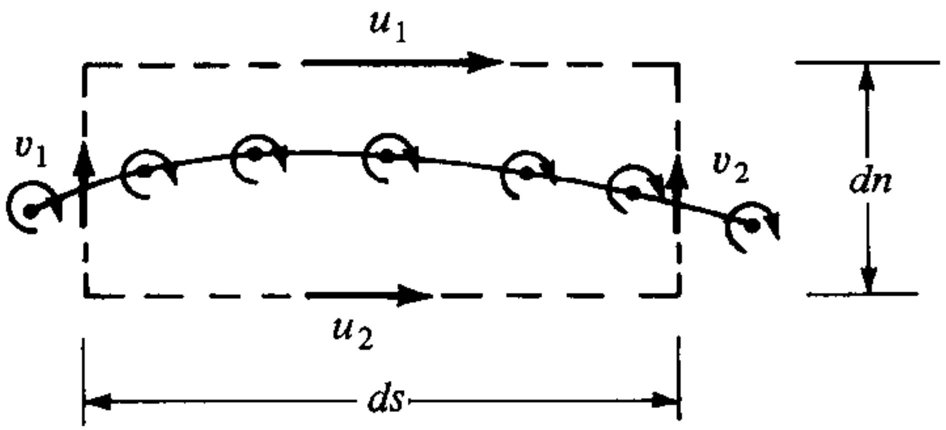


Figure 4.9 Tangential velocity jump across a vortex sheet.

Circulation around the dashed closed path is

$$\Gamma = \oint_C \vec{V} \cdot d\vec{s} = v_1 dn + u_1 ds - v_2 dn - u_2 ds$$

$$\Rightarrow \Gamma = (u_1 - u_2) ds \qquad ; \; dn \to 0$$

$$\Rightarrow \gamma \, ds = (u_1 - u_2) ds \qquad ; \; \gamma \text{ is the local sheet strength per unit length}$$

$$\boxed{\Rightarrow \gamma = u_1 - u_2}$$

Local jump in tangential velocity across the vortex sheet is equal to the local sheet strength.

Low Speed Flow Over Airfoils

The concept of vortex sheet is instrumental in the theoretical aerodynamics of low-speed airfoil.

A philosophy of airfoil theory for inviscid, incompressible flow is as follows:

> Consider an airfoil of arbitrary shape and thickness in a free stream with velocity V_∞ as sketched in the figure. **Replace the airfoil surface with a vortex sheet** of variable strength $\gamma(s)$. Calculate the variation of γ as a function of s such that the induced velocity field from the vortex sheet when added to the uniform velocity of magnitude V_∞ will make the vortex sheet (hence the airfoil surface) a **streamline of the flow**.

In turn, the circulation around the airfoil will given by

$$\Gamma = \int \gamma \, ds$$

where the integral is taken around the complete surface of the airfoil.

Finally, lift is calculated by the Kutta - Joukowski theorem :

$$L = \rho_\infty V_\infty \Gamma$$

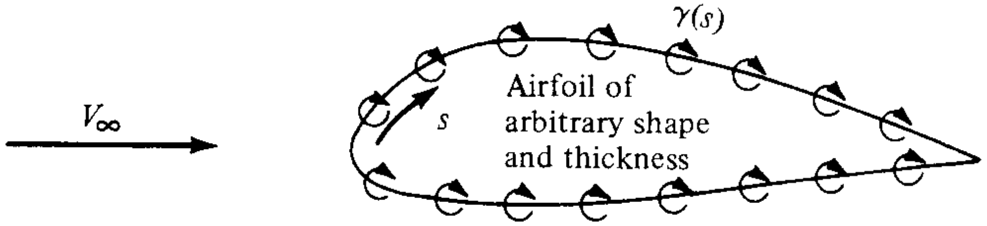


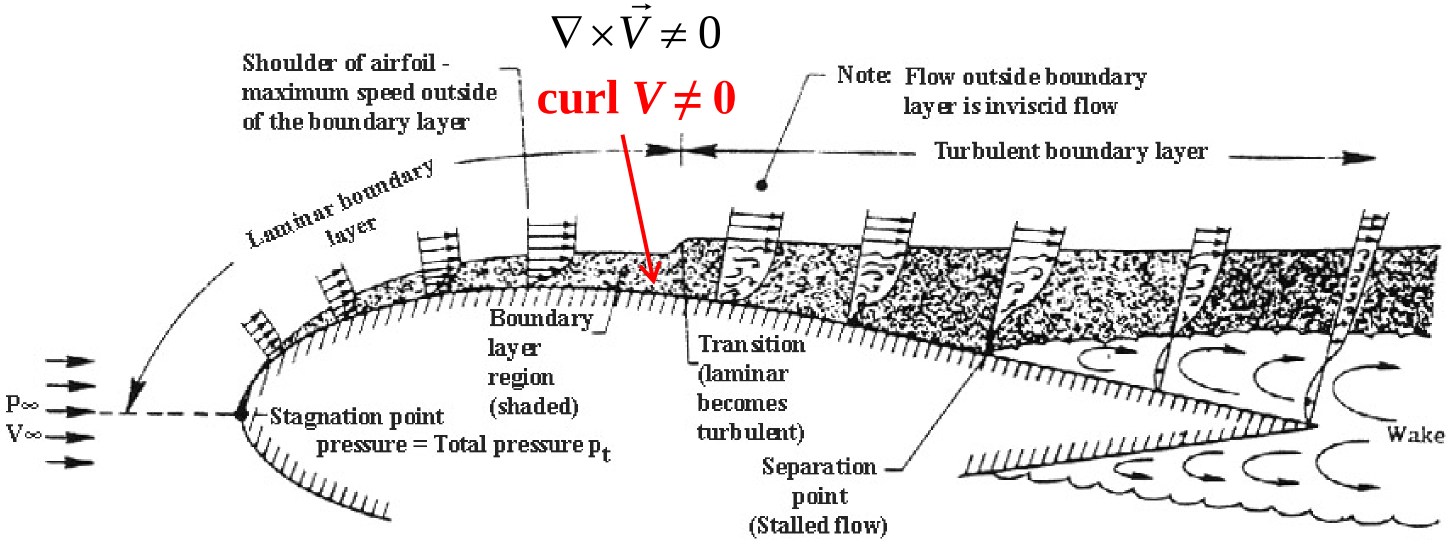
Figure 4.10 Simulation of an arbitrary airfoil by distributing a vortex sheet over the airfoil surface.

Low Speed Flow Over Airfoils

The concept of replacing the airfoil surface with a vortex sheet is more than just a mathematical device; it also has physical significance. In real case, there is a thin boundary layer on the surface, due to the action of friction between the surface and the air flow. This boundary layer is highly viscous region in which large velocity gradients produce substantial vorticity **(curl $V \neq 0$)**.

Hence, in real life, there is a distribution of vorticity along the airfoil surface due to viscous effect and the philosophy of replacing the airfoil surface with a vortex sheet can be constructed as a way of modeling this effect in an inviscid flow.

Modeling of boundary layer as vortices (vortex sheet):



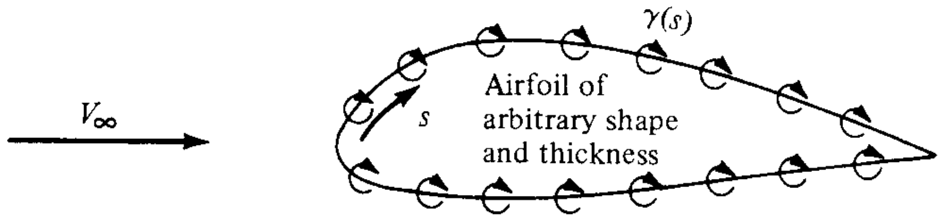


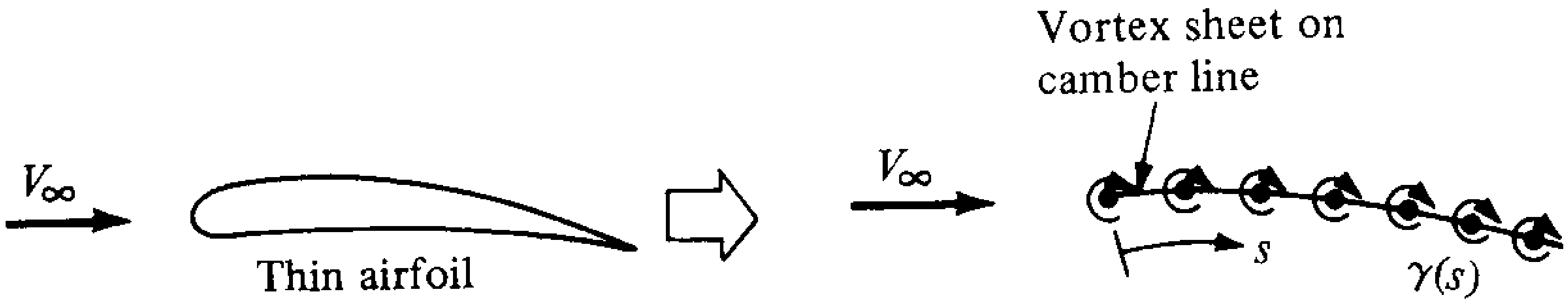
Figure 4.10 Simulation of an arbitrary airfoil by distributing a vortex sheet over the airfoil surface.

Low Speed Flow Over Airfoils

Imagine that the airfoil is made **very thin**. If we were to stand back and look at such a thin airfoil from a distance, the portion of the **vortex sheet on the top and bottom surface of the airfoil would almost coincide**.

This give rise to a method of approximating a **thin airfoil be replacing it with a single vortex sheet distributed over the camber line of the airfoil**, as shown in figure.

The strength of this vortex sheet is calculated such that, in combination with free stream, **the camber line becomes a streamline of the flow. This philosophy is known as classical thin airfoil theory.**



thin airfoil approximation

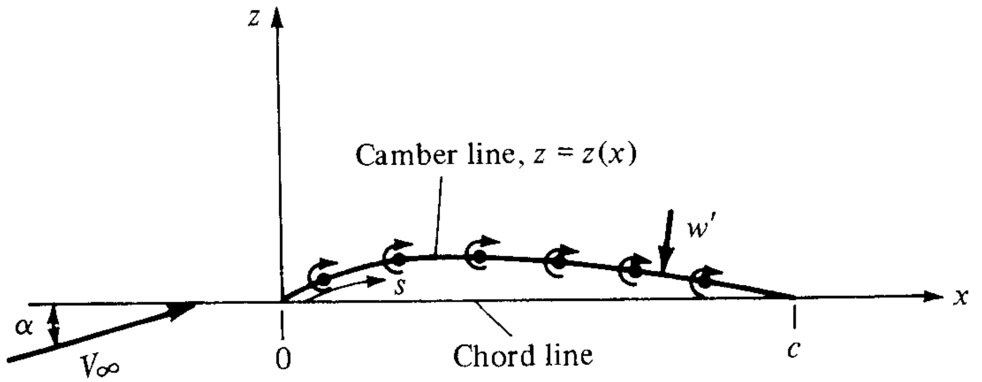
Thin Airfoil Theory

Consider a vortex sheet placed on the camber line of an airfoil, as shown in Fig. (a). The free-stream velocity is V_∞, and the airfoil is at an angle of attack (AOA), α. The distance measured along the camber line is denoted by *s*. The shape of the camber line is given by $z = z(x)$. The chord length is *c*.

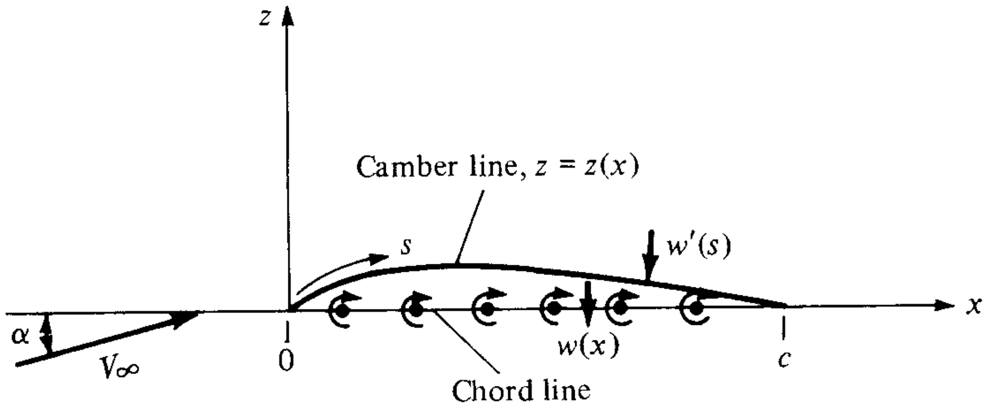
w' is the component of velocity normal to the camber line induced by the vortex sheet; $w' = w'(s)$

If the airfoil is thin, the camber line is close to the chord line, and viewed from a distance, the vortex sheet appears to fall approximately on the chord line as shown in Fig. (b).

Here $\gamma = \gamma(x)$ and $\gamma = \gamma(x)$ is calculated to satisfy that the camber line (not the chord line) is a streamline.



(*a*) Vortex sheet on the camber line



(*b*) Vortex sheet on the chord line

Figure 4.17 Placement of the vortex sheet for thin airfoil analysis.

Thin Airfoil Theory

For the camber line to be a streamline, the component of velocity normal to the camber line must be zero at all points along the camber line.

The velocity at any point in the flow is the sum of uniform velocity and the velocity induced by the vortex sheet.

Let $V_{\infty,n}$ be the component of free stream velocity normal to the camber line. Thus, for the camber line to be a streamline;

$$\boxed{V_{\infty,n} + w'(s) = 0}$$

At every point along the camber line.

At any point P on the camber line, where the slope of the camber line is dz/dx, the geometry of figures yields;

$$V_{\infty,n} = V_\infty \sin\left[\alpha + \tan^{-1}\left(-\frac{dz}{dx}\right)\right] \approx V_{\infty,n} = V_\infty\left(\alpha - \frac{dz}{dx}\right)$$

for a thin airfoil at small angle of attack α and $\tan^{-1}\left(-\frac{dz}{dx}\right) \rightarrow -\frac{dz}{dx}$

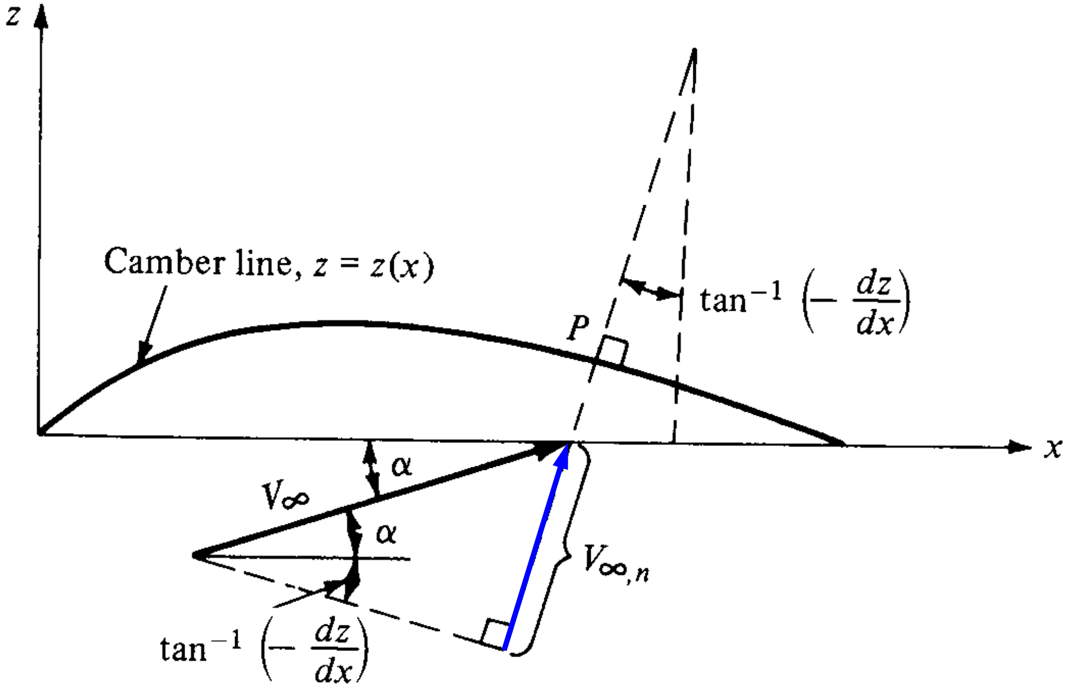
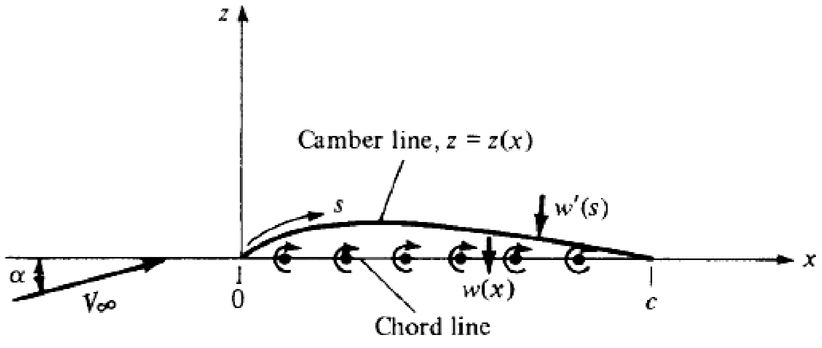


Figure 4.18 Determination of the component of freestream velocity normal to the camber line.

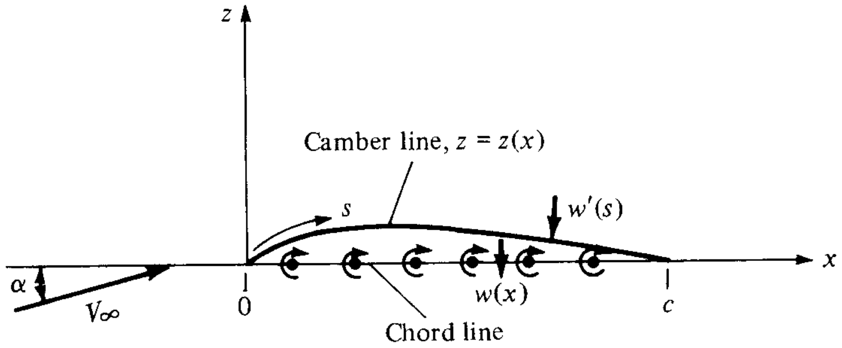


Thin Airfoil Theory

If the airfoil is thin, the camber line is close to the chord line, and it is consistent with thin airfoil theory to make the approximation that

$$w'(s) \approx w(x)$$

an expression for $w(x)$ in terms of the strength of the vortex sheet is easily obtainable as follows:



(b) Vortex sheet on the chord line

Figure 4.17 Placement of the vortex sheet for thin airfoil analysis.

Consider an elemental vortex of strength $\gamma d\xi$ located at a distance ξ from the origin along the chord line, as shown in figure. The strength of the vortex sheet γ varies with the distance along the chord line; that is $\gamma = \gamma(\xi)$. The velocity dw at point x induced by the elemental vortex at point ξ is:

$$dw = -\frac{\gamma(\xi)d\xi}{2\pi(x-\xi)}$$

γ = Vortex strength per unit length

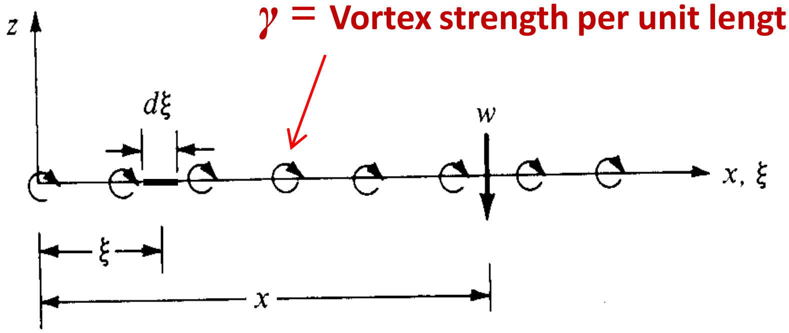


Figure 4.19 Calculation of the induced velocity at the chord line.

recall:

$$V_\theta = -\frac{\Gamma}{2\pi r} \text{ ; due to vortex (clockwise)}$$

Thin Airfoil Theory

In turn, the velocity $w(x)$ induced at point x by all the elemental vortices along the chord line is obtained by integrating from leading edge ($\xi = 0$) to the trailing edge ($\xi = c$) :

$$w(x) = -\int_0^c \frac{\gamma(\xi)d\xi}{2\pi(x-\xi)}$$

$$\approx w'(s) = -\int_0^c \frac{\gamma(\xi)d\xi}{2\pi(x-\xi)}$$

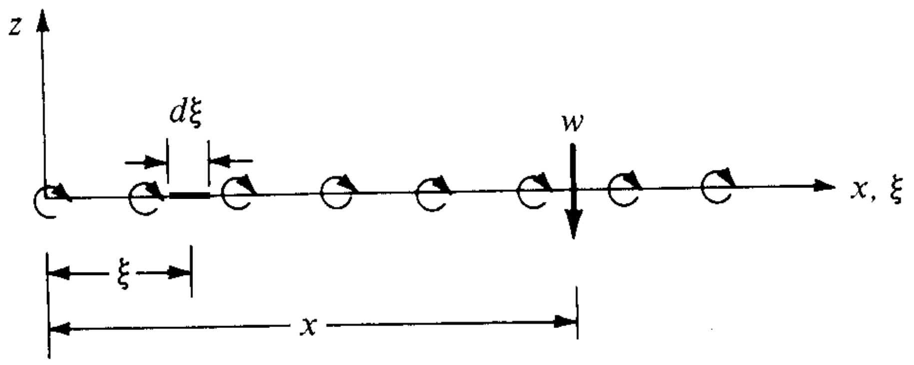


Figure 4.19 Calculation of the induced velocity at the chord line.

for the camberline to be a streamline,

$$V_{\infty,n} + w'(s) = 0$$

$$\Rightarrow V_\infty\left(\alpha - \frac{dz}{dx}\right) - \int_0^c \frac{\gamma(\xi)d\xi}{2\pi(x-\xi)} = 0$$

$$\boxed{\Rightarrow \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{(x-\xi)} = V_\infty\left(\alpha - \frac{dz}{dx}\right)}$$

(Integro-differential equation)

←

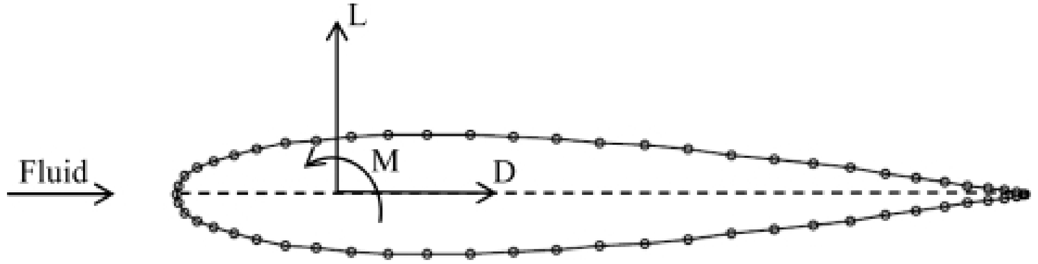
Fundamental equation of thin airfoil theory.

To be solved for symmetric airfoil and cambered airfoil.

Thin Airfoil Theory

Symmetric airfoil

A symmetric airfoil has no camber; the camber line is coincident with the chord line.



$$\frac{dz}{dx} = 0$$

$$\Rightarrow \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{(x-\xi)} = V_\infty\left(\alpha - \cancel{\frac{dz}{dx}}\right)$$

$$\Rightarrow \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{(x-\xi)} \;=\; V_\infty\alpha$$

To deal with the integral in above equation; adopt the following transformation:

$$\xi = \frac{c}{2}(1-\cos\theta) \qquad \therefore\ d\xi = \frac{c}{2}\sin\theta\, d\theta$$

$$\text{at L.E.}\quad \xi = 0 \quad ; \quad \theta = 0$$

$$\text{at T.E.}\quad \xi = c \quad ; \quad \theta = \pi$$

Further, since x is a fixed point in the above equation, it corresponds as

$$x = \frac{c}{2}(1-\cos\theta_0)$$

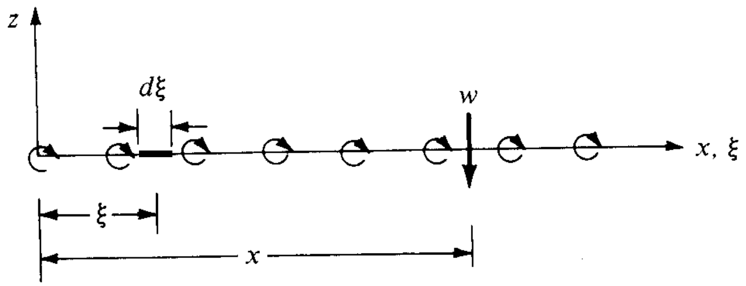
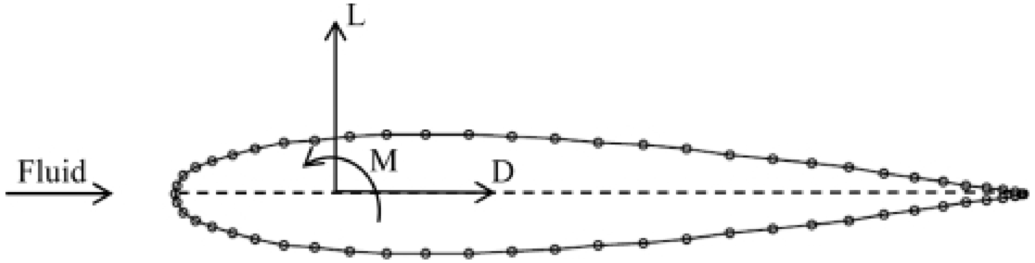


Figure 4.19 Calculation of the induced velocity at the chord line.

Thin Airfoil Theory



Substitute the above transformation gives:

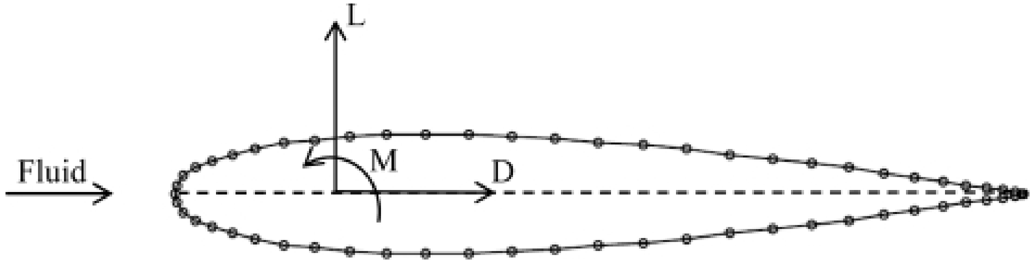
$$\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta)\sin\theta\, d\theta}{\cos\theta-\cos\theta_0} = V_\infty\alpha$$

A rigorous solution of the above equation can be obtained from the mathematical theory of integral equations (self-study)

$$\boxed{\gamma(\theta)=2\alpha V_\infty\frac{(1+\cos\theta)}{\sin\theta}}$$

Distribution of circulation around a symmetric airfoil

Thin Airfoil Theory



Now, the total circulation around the symmetric airfoil is

$$\Gamma = \int_0^c \gamma(\xi)\,d\xi$$

$$\Rightarrow \Gamma = \frac{c}{2}\int_0^\pi \gamma(\theta)\sin\theta\,d\theta \quad ; \quad \text{transformation, } \xi = \frac{c}{2}(1-\cos\theta)$$

$$\Rightarrow \Gamma = \frac{c}{2}\int_0^\pi 2\alpha V_\infty \frac{(1+\cos\theta)}{\sin\theta}\sin\theta\,d\theta$$

$$\boxed{\because \gamma(\theta) = 2\alpha V_\infty \frac{(1+\cos\theta)}{\sin\theta}}$$

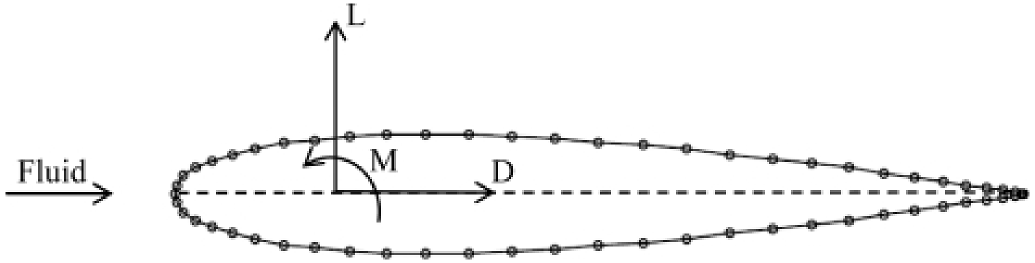
$$\Rightarrow \Gamma = \alpha c V_\infty \int_0^\pi (1+\cos\theta)\,d\theta$$

$$\boxed{\Rightarrow \Gamma = \pi\alpha c V_\infty} \longleftarrow$$

Thin Airfoil Theory

In case of **symmetric airfoil**;

$$\Gamma = \pi \alpha c V_\infty$$



Using Kutta-Joukowski theorem;

$$L' = \rho_\infty V_\infty \Gamma \qquad \text{(lift per unit span of airfoil/sectional lift)}$$

$$\Rightarrow L' = \pi \alpha c \rho_\infty V_\infty{}^2 \qquad (\because \;\; \Gamma = \pi \alpha c V_\infty)$$

The sectional lift coefficient

$$c_l = \frac{L'}{\frac{1}{2}\rho_\infty V_\infty{}^2 c} = \frac{\pi \alpha c \rho_\infty V_\infty{}^2}{\frac{1}{2}\rho_\infty V_\infty{}^2 c}$$

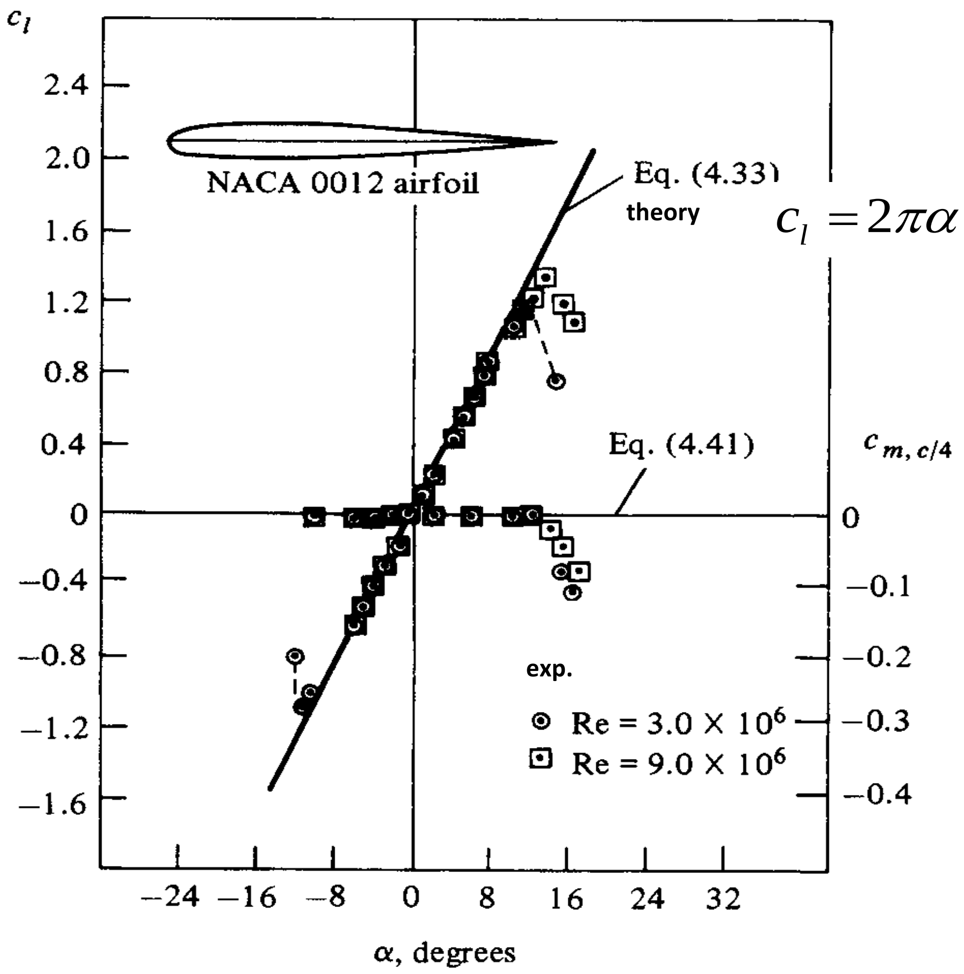
$$\boxed{\Rightarrow c_l = 2\pi\alpha}$$

⟵ **Theoretical lift coefficient is linearly proportional to the angle of attack (AOA).**

$$\boxed{\text{lift slope} = \frac{dc_l}{d\alpha} = 2\pi}$$

⟵ **Theoretical lift slope is 2π per radian, which is 0.11 per degree of AOA.**

Thin Airfoil Theory



$$c_l = 2\pi\alpha$$

Figure 4.20 Comparison between theory and experiment for the lift and moment coefficients for an NACA 0012 airfoil. *(Source: Abbott and von Doenhoff, Reference 11.)*

- Theory can’t predict the stalling phenomena.
- Theory can’t differential the effect of Re number.
- Theory can’t accommodate the thickness of the airfoil.

Asymmetric / Cambered Airfoil

Thin Airfoil Theory

According to thin airfoil theory--

The strength of the vortex sheet is calculated such that, in combination with free stream, **the camber line becomes a streamline of the flow.** **This philosophy is known as classical thin airfoil theory.**

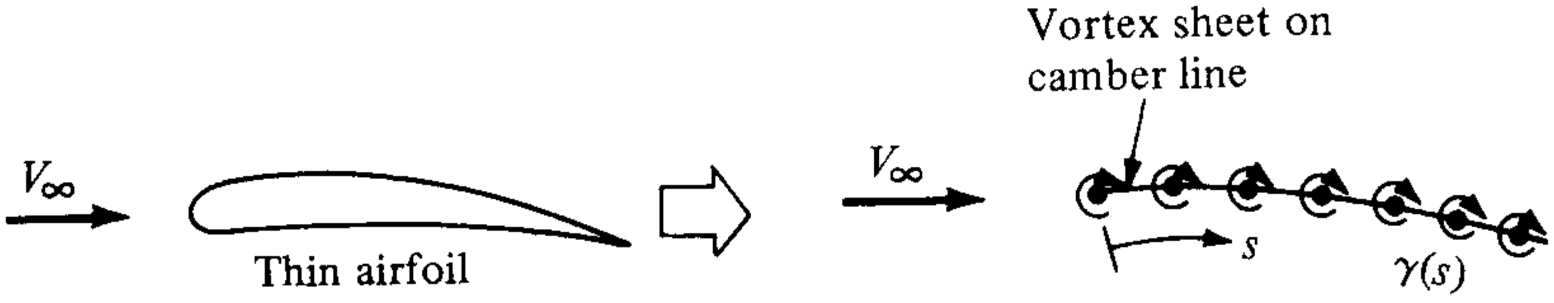


Figure 4.11 Thin airfoil approximation.

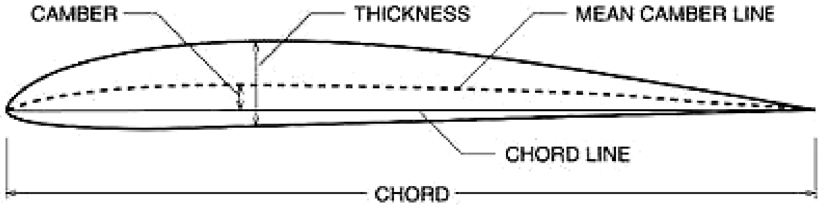
Fundamental equation of thin airfoil theory is

$$\boxed{\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{(x-\xi)} = V_\infty\left(\alpha - \frac{dz}{dx}\right)}$$

Thin Airfoil Theory

Asymmetric / Cambered airfoil

This type of airfoil has a finite amount of camber i.e.:



$$\frac{dz}{dx} \neq 0$$

$$\Rightarrow \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{(x-\xi)} = V_\infty\left(\alpha - \frac{dz}{dx}\right) \qquad \text{(i)}$$

To deal with the integral in above equation; adopt the following transformation:

$$\xi = \frac{c}{2}(1-\cos\theta) \qquad \therefore \quad d\xi = \frac{c}{2}\sin\theta\, d\theta$$

$$\text{at LE} \quad \xi = 0 \quad ; \quad \theta = 0$$

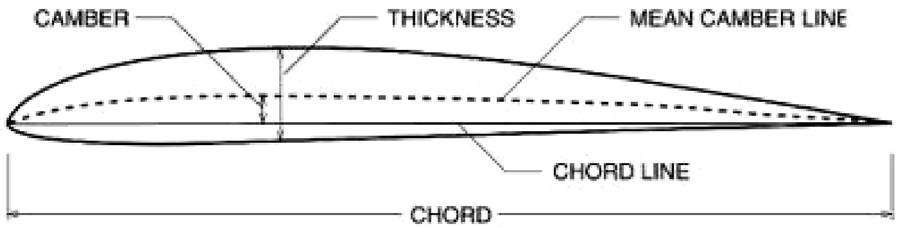
$$\text{at TE} \quad \xi = c \quad ; \quad \theta = \pi$$

Further, since *x* is a fixed point in the above equation, it corresponds as

$$x = \frac{c}{2}(1-\cos\theta_0)$$

Thin Airfoil Theory

Substitute the above transformation in equation (i), gives:



$$\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta)\sin\theta\,d\theta}{\cos\theta-\cos\theta_0} = V_\infty\left(\alpha-\frac{dz}{dx}\right) \qquad \text{(ii)}$$

Solution of the above equation from mathematical theory of integral equations gives as

$$\boxed{\gamma(\theta)=2V_\infty\left(A_0\frac{1+\cos\theta}{\sin\theta}+\sum_{n=1}^{\infty}A_n\sin n\theta\right) \qquad \text{(iii)}}$$

Distribution of circulation around asymmetric/cambered airfoil

Note that the above expression consists of a leading term very similar to the expression for the case of symmetric airfoil, **plus a Fourier sine series with coefficients A_n**. The values of A_n depend on the shape of the camber line *dz/dx*, and A_0 depends on both *dz/dx* and α.

$$\boxed{\gamma(\theta)=2\alpha V_\infty\frac{(1+\cos\theta)}{\sin\theta}}$$

Distribution of circulation around a symmetric airfoil

Thin Airfoil Theory

To find the specific values of A_0 and A_n (n =1,2,3,….), substitute expression (iii) in equation (ii)

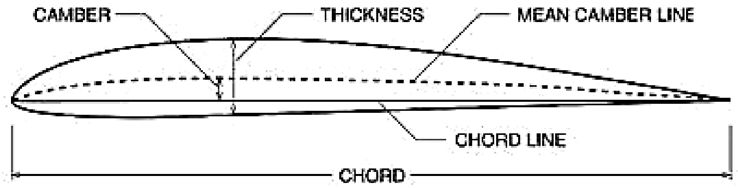
$$\frac{1}{\pi}\int_0^{\pi}\frac{A_0(1+\cos\theta)}{\cos\theta-\cos\theta_0}d\theta+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi}\frac{A_n\sin n\theta\,\sin\theta}{\cos\theta-\cos\theta_0}d\theta=\alpha-\frac{dz}{dx}\qquad\text{(iv)}$$

The first integral can be evaluated from the standard form of integration:

$$\boxed{\int_0^{\pi}\frac{\cos n\theta}{\cos\theta-\cos\theta_0}d\theta=\frac{\pi\sin n\theta_0}{\sin\theta_0}\qquad\text{(v)}}\quad\longrightarrow\quad\boxed{n=1;\quad\int_0^{\pi}\frac{\cos\theta}{\cos\theta-\cos\theta_0}d\theta=\pi}$$

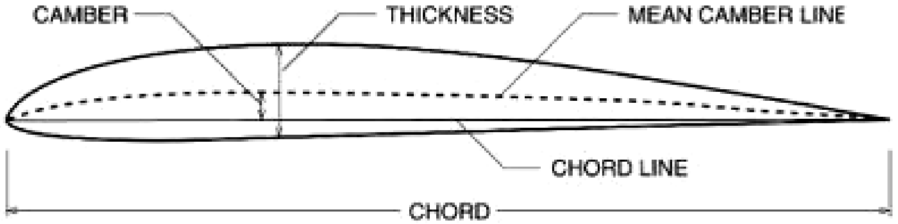
The remaining integral can be obtained from

$$\boxed{\int_0^{\pi}\frac{\sin n\theta\sin\theta}{\cos\theta-\cos\theta_0}d\theta=-\pi\cos n\theta_0\qquad\text{(vi)}}$$



Thin Airfoil Theory

Using the above expressions (v) and (vi) in equation (iv)



$$A_0 - \sum_{n=1}^{\infty} A_n \cos n\theta_0 = \alpha - \frac{dz}{dx}$$

$$\Rightarrow \frac{dz}{dx} = (\alpha - A_0) + \sum_{n=1}^{\infty} A_n \cos n\theta_0 \qquad \text{(vii)}$$

It is the form of **Fourier cosine series expansion for the function of dz/dx**.

In general, the **Fourier cosine series** representation of a function $f(\theta)$ over an interval $0 \le \theta \le 2\pi$

$$f(\theta) = B_0 + \sum_{n=1}^{\infty} B_n \cos n\theta$$

where

$$B_0 = \frac{1}{\pi}\int_0^{\pi} f(\theta)\, d\theta$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta \, d\theta$$

Thin Airfoil Theory

$$\boxed{\Rightarrow \frac{dz}{dx} = (\alpha - A_0) + \sum_{n=1}^{\infty} A_n \cos n\theta_0 \qquad \text{(vii)}}$$

Thus the coefficients of equation (vii) are;

$$\frac{dz}{dx} = \underbrace{(\alpha - A_0)}_{B_0} + \sum_{n=1}^{\infty} \underbrace{A_n}_{B_n} \cos n\theta_0 \qquad \text{(vii)}$$

A_0 depends on both α and the shape of the camber, dz/dx

$$(\alpha - A_0) = \frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}\, d\theta$$

$$\boxed{\Rightarrow A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}\, d\theta}$$

A_n depend on only on the shape of the camber, dz/dx

$$\boxed{A_n = \frac{2}{\pi}\int_0^{\pi} \frac{dz}{dx} \cos n\theta \, d\theta}$$

Thin Airfoil Theory

Now, total circulation due to entire vortex sheet from LE to TE is

$$\Gamma = \int_0^c \gamma(\xi)\, d\xi = \frac{c}{2}\int_0^\pi \gamma(\theta)\sin\theta\, d\theta \quad [\text{using the transformation}: \ \xi = \frac{c}{2}(1-\cos\theta)]$$

$$\Rightarrow \ \Gamma = cV_\infty\left[A_0\int_0^\pi (1+\cos\theta)\, d\theta + \sum_{n=1}^{\infty} A_n \int_0^\pi \sin n\theta \sin\theta\, d\theta\right] \qquad \text{(vii)}$$

$$\boxed{\gamma(\theta) = 2V_\infty\left(A_0\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right) \qquad \text{(iii)}}$$

Now, from standard table of integrals:

$$\int_0^\pi (1+\cos\theta)\, d\theta = \pi$$

and
$$\int_0^\pi \sin n\theta \sin\theta\, d\theta = \begin{cases} \dfrac{\pi}{2} & \text{for } n=1 \\ 0 & \text{for } n\neq 1 \end{cases}$$

So, the equation (vii) comes as

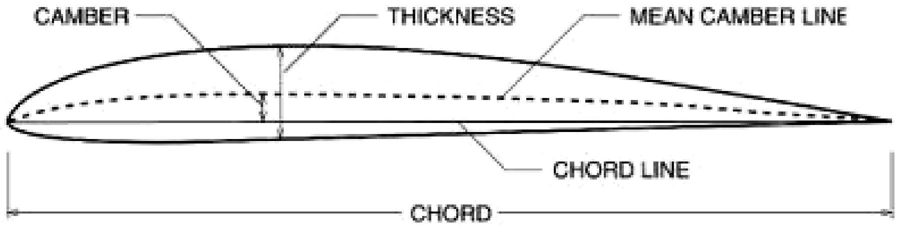
$$\boxed{\Rightarrow \ \Gamma = cV_\infty\left(\pi A_0 + \frac{\pi}{2}A_1\right) \qquad \text{(viii)}}$$

$$\boxed{A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}\, d\theta}$$

$$\boxed{A_1 = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos\theta\, d\theta}$$

Thin Airfoil Theory

Using **Kutta-Joukowski theorem**, Lift per unit span/sectional lift is;



$$L' = \rho_\infty V_\infty \Gamma$$

$$\boxed{\Rightarrow \quad L' = \rho_\infty V_\infty^{\;2} c \left(\pi A_0 + \frac{\pi}{2} A_1 \right) \qquad (\text{ix})}$$

And the lift coefficient,

$$c_l = \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^{\;2}(c)}$$

$$\Rightarrow \quad c_l = \pi\left(2A_0 + A_1\right) \qquad (x)$$

$$\boxed{A_0 = \quad \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}\, d\theta}$$

$$\boxed{A_1 = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos\theta \; d\theta}$$

$$\boxed{\Rightarrow \quad c_l = 2\pi\left[\alpha + \underline{\frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta - 1)d\theta}\right] \qquad (xi)}$$

Additional term due to camber

And the lift slope is,

$$\boxed{\Rightarrow \quad \frac{dc_l}{d\alpha} = 2\pi \qquad (xii)}$$

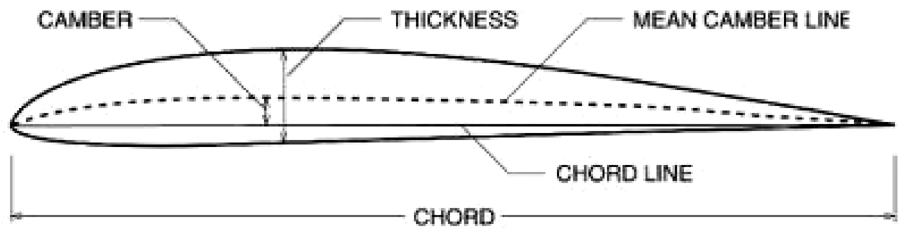
Gives the same result irrespective of magnitude of camber in asymmetric/cambered airfoil !!!

Thin Airfoil Theory

From the lift curve;

$$c_l = \frac{dc_l}{d\alpha}\left(\alpha - \alpha_{L=0}\right)$$

$$\Rightarrow \quad c_l = 2\pi\left(\alpha - \alpha_{L=0}\right) \qquad \because \text{lift slope, } \frac{dc_l}{d\alpha} = 2\pi \qquad (xiii)$$

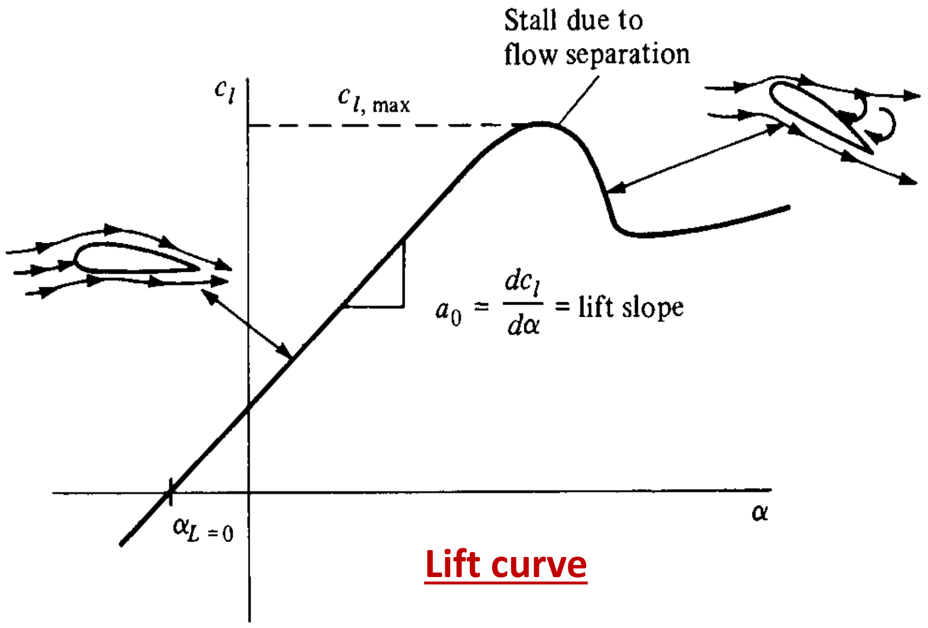


$$\boxed{\Rightarrow \quad c_l = 2\pi\left[\alpha + \frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}(\cos\theta - 1)\right]d\theta \qquad (xi)}$$

Comparing equation (xi) and (xiii) gives:

$$\boxed{\alpha_{L=0} = -\frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}(\cos\theta - 1)\; d\theta \qquad (xiv)}$$

This is the expression to determine the angle of attack (AOA) at zero lift.



Lift curve

Moment

$$M'_{\mathrm{LE}} = -\int_0^c \xi(dL) = -\rho_\infty V_\infty \int_0^c \xi\gamma(\xi)\,d\xi \qquad (4.35)$$

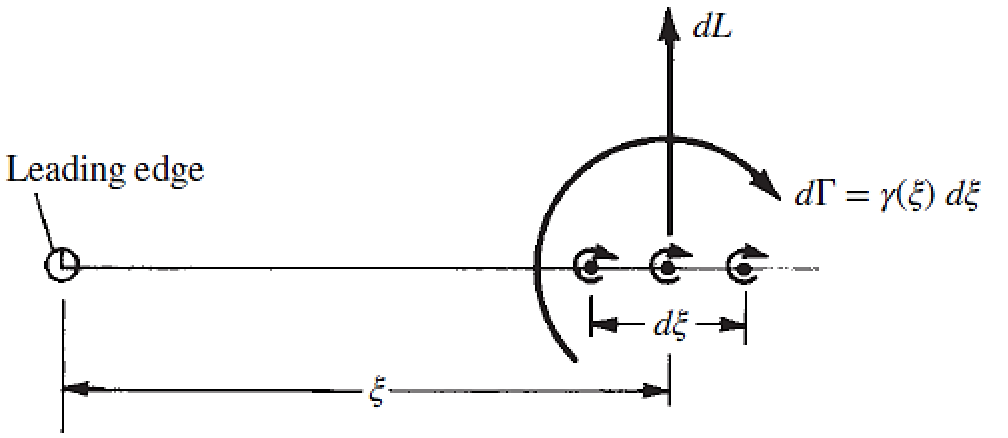


Figure 4.26 Calculation of moments about the leading edge.

The moment coefficient is

$$c_{m,\mathrm{le}} = \frac{M'_{\mathrm{LE}}}{q_\infty Sc}$$

$$\boxed{c_{m,c/4} = \frac{\pi}{4}(A_2 - A_1)} \qquad (4.64)$$

Thin Airfoil Theory

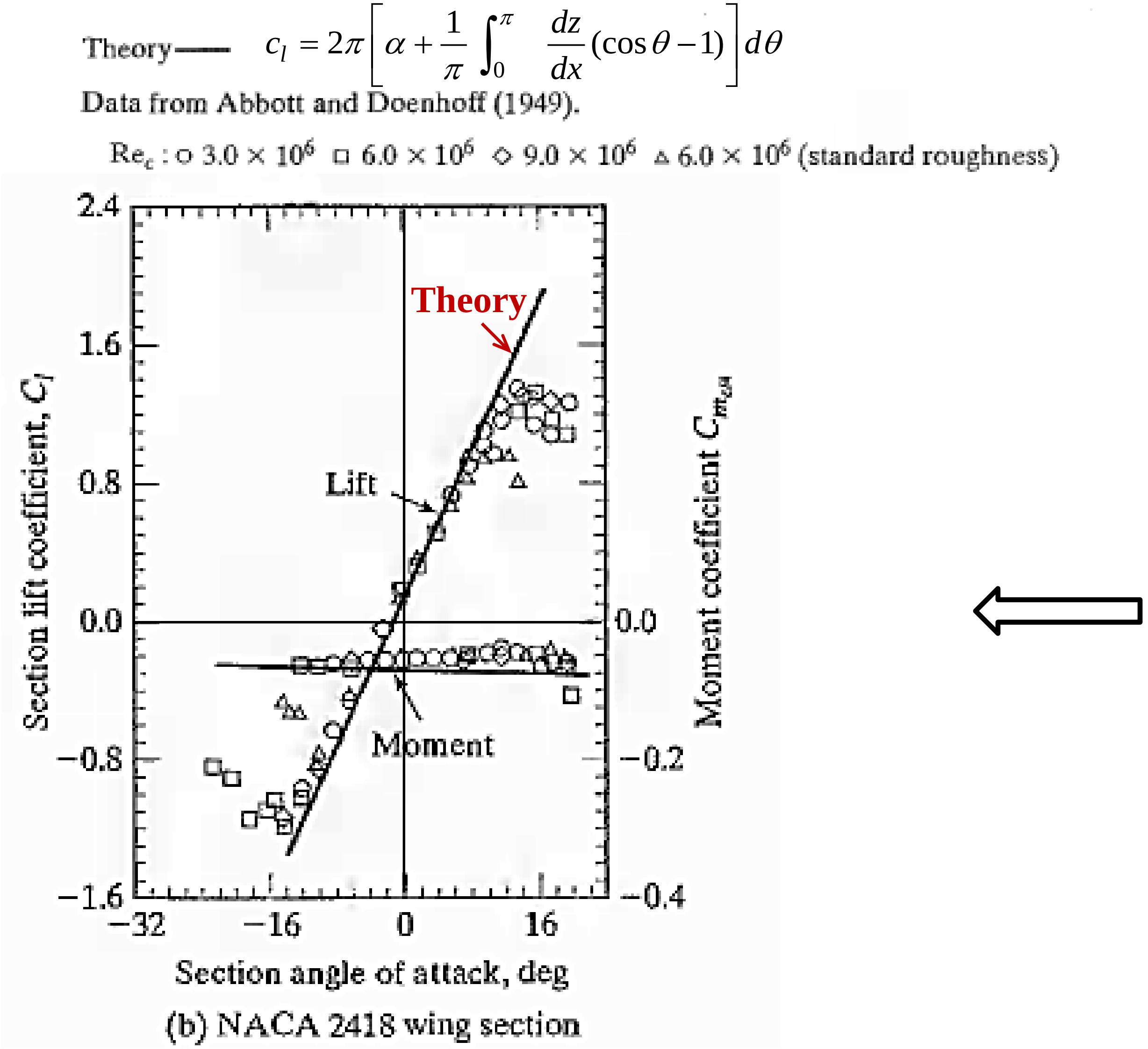
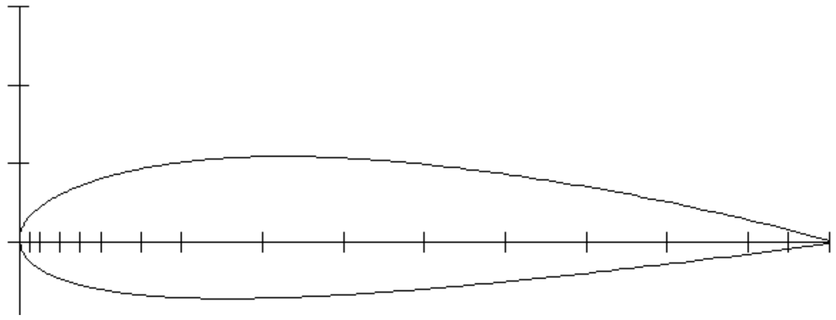


Fig. Lift curve for a cambered airfoil (NACA 2418)



NACA 2418 airfoil
(asymmetric airfoil)

Problem

Problem (Anderson Ex. 4.6)

Consider an NACA 23012 airfoil. The mean camber line for this airfoil is given by

$$\frac{z}{c} = 2.6595\left[\left(\frac{x}{c}\right)^3 - 0.6075\left(\frac{x}{c}\right)^2 + 0.1147\left(\frac{x}{c}\right)\right] \qquad \text{for } 0 \le \frac{x}{c} \le 0.2025$$

and

$$\frac{z}{c} = 0.02208\left(1 - \frac{x}{c}\right) \qquad \text{for } 0.2025 \le \frac{x}{c} \le 1.0$$

Calculate

(a) The angle of attack at zero lift

(b) The lift coefficient when $\alpha = 4°$